Towards Automatic Proofs of Inequalities Involving Elementary Functions

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Introduction and Motivation

- There are many applications in mathematics and engineering where proofs involving functions such as ln, exp, sin, cos, etc. are required.
- In their formalization of the Prime Number Theorem, Avigad and his colleagues (from CMU), spent much time proving simple facts involving logarithms.
- Other applications are not difficult to find.

Introduction and Motivation, Cont'd

- Our Starting Point: The theory of Real Closed Fields (RCF) - that is, the real numbers with addition and multiplication - is decidable.
- However: Inequalities involving elementary functions lie outside the scope of decision procedures, and can only be solved using heuristic methods. (Richardson's Theorem)
- Our Idea: Replace each occurrence of an elementary function by an upper or lower bound, as appropriate. Then, supply the reduced algebraic inequality problem to a decision procedure for the theory of real closed fields (RCF).

Related Work

- Tarski found the first quantifier elimination procedure which solves problems over the reals involving + - * / in the 1930s.
- Collins introduced the first feasible method (cylindrical algebraic decomposition) in 1975.
- One freely-available implementation is the QEPCAD decision procedure.
- HOL Light provides *REAL_QELIM_CONV* and *REAL_SOS*.
- Other heuristic procedures such as Hunt et. al. and Tiwari.
- Mũnoz and Lester's method is based on upper and lower bounds for the elementary functions, coupled with interval arithmetic.

Families of Lower and Upper Bounds

Functions $\underline{f} : (\mathbb{R}, \mathbb{N}) \rightarrow \mathbb{R}$ and $\overline{f} : (\mathbb{R}, \mathbb{N}) \rightarrow \mathbb{R}$ are closed under $\mathbb{Q}$ such that:

$$\underline{f}(x, n) \leq f(x) \leq \overline{f}(x, n),$$

$$\underline{f}(x, n) \leq \underline{f}(x, n+1)$$

$$\overline{f}(x, n+1) \leq \overline{f}(x, n)$$

$$\lim_{x \to \infty} \underline{f}(x, n) = f(x) = \lim_{x \to \infty} \overline{f}(x, n)$$

Bounds for the Exponential Function

$$\underline{\exp}(x,n) = \sum_{i=0}^{2(n+1)+1} \frac{x^i}{i!} \qquad \text{if } -1 \le x < 0$$

$$\overline{\exp}(x,n) = \sum_{i=0}^{2(n+1)} \frac{x^i}{i!} \qquad \text{if } -1 \le x < 0$$

$$\underline{\exp}(0,n) = \overline{\exp}(0,n) = 1$$

$$\underline{\exp}(x,n) = \frac{1}{\overline{\exp}(-x,n)} \qquad \text{if } 0 < x \le 1$$

$$\overline{\exp}(x,n) = \frac{1}{\underline{\exp}(-x,n)} \qquad \text{if } 0 < x \le 1$$

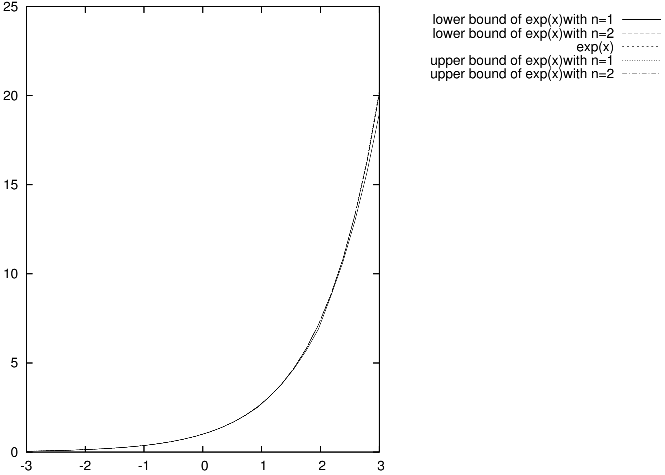
Bounds for the Exponential Function, Cont'd

$$\underline{\exp}(x, n) = \underline{\exp}(x/m, n)^m \quad \text{if} \;\; x < -1,\; m = -\lfloor x \rfloor$$

$$\overline{\exp}(x, n) = \overline{\exp}(x/m, n)^m \quad \text{if} \;\; x < -1,\; m = -\lfloor x \rfloor$$

$$\underline{\exp}(x, n) = \overline{\exp}(x/m, n)^m \quad \text{if} \;\; 1 < x,\; m = \lfloor -x \rfloor$$

$$\overline{\exp}(x, n) = \underline{\exp}(x/m, n)^m \quad \text{if} \;\; 1 < x,\; m = \lfloor -x \rfloor$$


25
20
15
10
5
0
-3
-2
-1
0
1
2
3
lower bound of exp(x)with n=1
lower bound of exp(x)with n=2
exp(x)
upper bound of exp(x)with n=1
upper bound of exp(x)with n=2

Bounds for the Logarithmic Function

$$\underline{\ln}(x,n) = \sum_{i=1}^{2n} (-1)^{i+1} \frac{(x-1)^i}{i} \quad \text{if } 1 < x \leq 2$$

$$\overline{\ln}(x,n) = \sum_{i=1}^{2n+1} (-1)^{i+1} \frac{(x-1)^i}{i} \quad \text{if } 1 < x \leq 2$$

$$\underline{\ln}(1,n) = \overline{\ln}(1,n) = 0$$

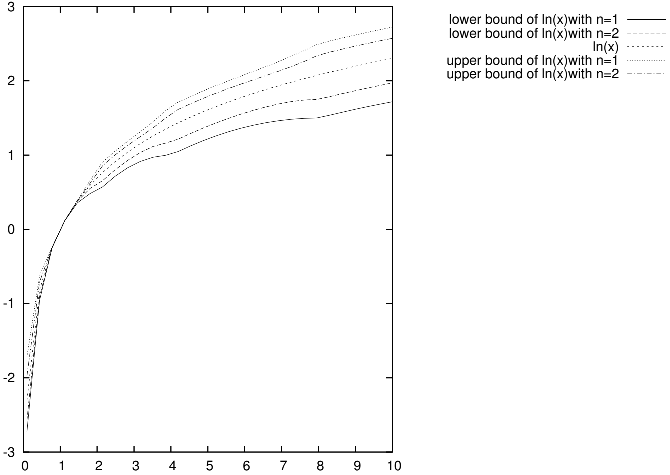
$$\underline{\ln}(x,n) = -\,\overline{\ln}\left(\frac{1}{x},\ n\right), \qquad \text{if } 0 < x < 1$$

$$\overline{\ln}(x,n) = -\,\underline{\ln}\left(\frac{1}{x},\ n\right), \qquad \text{if } 0 < x < 1$$

Bounds for the Logarithmic Function, Cont'd

$$\underline{\ln}(x, n) = m\,\underline{\ln}(2, n) + \underline{\ln}(y, n) \quad \text{if } x > 2,\ x = 2^m y,\ 1 < y \leq 2$$

$$\overline{\ln}(x, n) = m\,\overline{\ln}(2, n) + \overline{\ln}(y, n) \quad \text{if } x > 2,\ x = 2^m y,\ 1 < y \leq 2$$


lower bound of ln(x)with n=1
lower bound of ln(x)with n=2
ln(x)
upper bound of ln(x)with n=1
upper bound of ln(x)with n=2
3
2
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-1
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0
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6
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10

A Simple Example Concerning Exponentials

- Main Goal:

$$0 \leq x \leq 1 \Longrightarrow \exp x \leq 1 + x + x^2.$$

- It suffices to prove this algebraic formula:

$$0 \leq x \leq 1 \Longrightarrow \overline{\exp}\,(x, n) \leq 1 + x + x^2$$

- Case Analysis:

$$x = 0 \quad \text{or} \quad 0 < x \leq 1$$

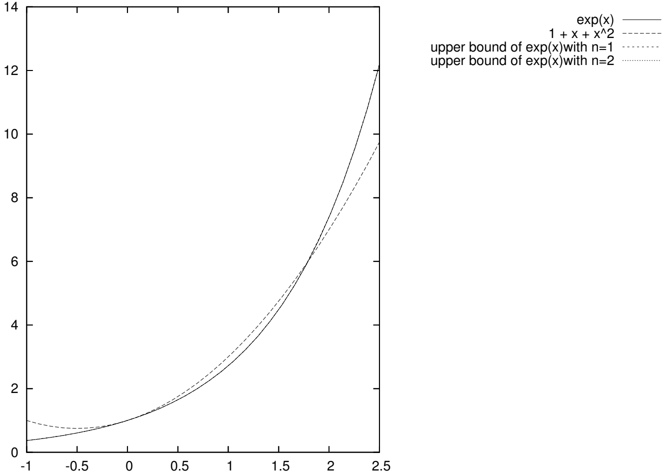
A Simple Example Concerning Exponentials, Cont'd

- If $x = 0$ then $\overline{\exp}(0, n) = 1 \leq 1 + 0 + 0^2 = 1$, trivially.
- If $0 < x \leq 1$, then

$$\overline{\exp}(x, n) = \left(\sum_{i=0}^{2(n+1)+1} \frac{(-x)^i}{i!} \right)^{-1}$$

Setting $n = 0$, it suffices to prove

$$\left(1 + (-x) + \frac{(-x)^2}{2} + \frac{(-x)^3}{6} \right)^{-1} \leq 1 + x + x^2.$$


14
12
10
8
6
4
2
0
-1
-0.5
0
0.5
1
1.5
2
2.5
exp(x)
1 + x + x^2
upper bound of exp(x)with n=1
upper bound of exp(x)with n=2

An Extended Example Concerning Logarithms

- Main Goal:

$$-\frac{1}{2} < x \leq 3 \Longrightarrow \ln(1 + x) \leq x.$$

- It suffices to prove this algebraic formula:

$$\frac{1}{2} < 1 + x \leq 4 \Longrightarrow \overline{\ln}(1 + x, n) \leq x$$

- Case Analysis:

$$\frac{1}{2} < 1{+}x < 1 \quad \text{or} \quad 1{+}x = 1 \quad \text{or} \quad 1 < 1{+}x \leq 2 \quad \text{or} \quad 2 < 1{+}x \leq 4$$

An Extended Example Concerning Logarithms, Cont'd

- If $1 + x = 1$, then $x = 0$ and $\overline{\ln}(1 + x, n) = \overline{\ln}(1, n) = 0 \le x$.
- If $1 < 1 + x \le 2$, then

$$\overline{\ln}(1 + x, n) = \sum_{i=1}^{2n+1} (-1)^{i+1} \frac{((1 + x) - 1)^i}{i} = \sum_{i=1}^{2n+1} (-1)^{i+1} \frac{x^i}{i}$$

Setting $n = 0$ yields $\overline{\ln}(1 + x, n) = x$ and reduces our inequality to the trivial $x \le x$.

An Extended Example Concerning Logarithms, Cont'd

- If $2 < 1 + x \leq 4$, then we need to find a positive integer m and some y such that $1 + x = 2^m y$ and $1 < y \leq 2$. Clearly $m = 1$. In this case, putting $n = 0$, we have

$$\begin{aligned}
\sum_{i=1}^{2n+1} (-1)^{i+1} \frac{(2-1)^i}{i} + \sum_{i=1}^{2n+1} (-1)^{i+1} \frac{(y-1)^i}{i} &= 1 + (y-1) \\
&= y \\
&\leq 2y - 1 \\
&= x.
\end{aligned}$$

An Extended Example Concerning Logarithms, Cont'd

- If $\frac{1}{2} < 1 + x < 1$, then $1 < 1/(1+x) < 2$. Putting $n = 1$, we have

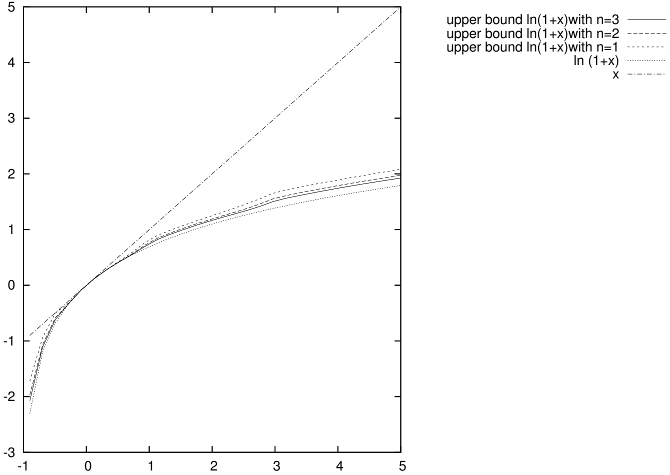
$$
\begin{aligned}
\overline{\ln}(1+x, n) &= -\underline{\ln}\left(\frac{1}{1+x}, n\right) \\
&= -\sum_{i=1}^{2n} (-1)^{i+1} \frac{\left(\frac{1}{1+x} - 1\right)^i}{i} \\
&= \sum_{i=1}^{2n} \frac{(-1)^i}{i} \left(\frac{-x}{1+x}\right)^i \\
&= \left(\frac{x}{1+x}\right) + \left(\frac{1}{2}\right)\left(\frac{-x}{1+x}\right)^2.
\end{aligned}
$$

An Extended Example Concerning Logarithms, Cont'd

Now

$$\begin{aligned}
\left(\frac{x}{1+x}\right) + \left(\frac{1}{2}\right)\left(\frac{-x}{1+x}\right)^2 \leq x &\iff \\
x(1+x) + \frac{1}{2}x^2 \leq x(1+x)^2 &\iff \\
x + \frac{3}{2}x^2 \leq x + 2x^2 + x^3 &\iff \\
-\frac{1}{2}x^2 \leq x^3 &\iff \\
-\frac{1}{2} \leq x &
\end{aligned}$$

which holds because $\frac{1}{2} < 1 + x$.


upper bound ln(1+x)with n=3
upper bound ln(1+x)with n=2
upper bound ln(1+x)with n=1
ln (1+x)
x
5
4
3
2
1
0
-1
-2
-3
-1
0
1
2
3
4
5

Logarithmic Problems

$$-\frac{1}{2} \leq x \leq 3 \Longrightarrow \frac{x}{1+x} \leq \ln(1+x) \leq x$$

$$0 \leq x \leq 3 \Longrightarrow |\ln(1+x) - x| \leq x^2$$

$$|x| \leq \frac{1}{2} \Longrightarrow |\ln(1+x) - x| \leq 2x^2$$

$$0 \leq x \leq 0.5828 \Longrightarrow |\ln(1-x)| \leq \frac{3x}{2}$$

Exponential Problems

$$0 \le x \le 1 \Longrightarrow e^{(x-x^2)} \le 1 + x$$

$$-1 \le x \le 1 \Longrightarrow 1 + x \le e^x$$

$$-1 \le x \le 1 \Longrightarrow e^x \le \frac{1}{1-x}$$

$$-\frac{1}{2} \le x \Longrightarrow e^{x/(1+x)} \le 1 + x$$

$$-1 \le x \le 0 \Longrightarrow e^x \le 1 + \frac{x}{2}$$

$$0 \le |x| \le 1 \Longrightarrow \frac{1}{4}|x| \le |e^x - 1| \le \frac{7}{4}|x|$$

Conclusions

- Our preliminary investigations are promising.
- We have used the method described above to solve about 30 problems.
- We manually reduced each problem to algebraic form, then tried to solve the reduced problems using three different tools.
- QEPCAD solved all of the problems, usually taking less than one second.
- HOL Light's sum-of-squares tool *REAL_SOS* solved all of the problems but two, again usually in less than a second.
- HOL Light's quantifier elimination tool *REAL_QELIM_CONV* solved all of the problems but three. It seldom required more than five seconds

Future Works

- Much work remains to be done before this procedure can be automated.
- We need to experiment with a variety of upper and lower bounds.
- Case analyses will still be inevitable, so we need techniques to automate them in the most common situations.
- We have to evaluate different ways of deciding the RCF problems that are finally generated.