Producing Conflict Sets for Combinations of Theories

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Outline



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Satisfiability Modulo Theory (SMT) Tools

- Efficient (in practice) tools to check the satisfiability of arbitrary Boolean combinations of ground atoms modulo a background theory T
- Based on the integration of a Boolean solver (used to enumerate Boolean assignments) and a decision procedure for the background theory T (used to prune unsatisfiable assignments wrt. T)
- Efficiency obtained (among other techniques) by minimizing the number of calls to the decision procedure for T ... *How?*
- The decision procedure for T is required to compute ***explanations*** of the unsatisfiability of the input set S of ground literals
  - *explanations* of the unsatisfiability of S = "*small*" subset of S which is still unsatisfiable in T

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Computing Conflict Sets

- ***QUESTION***: How to lift decision procedures solving the satisfiability problem in a theory T to also return conflict sets*?*
- ***PARTIAL ANSWER***: Some proposals ***[Fon04, dMRS04, NO05, ST05]*** when T is a single theory (T = "*theory of uninterpreted function symbols*")
- ***STILL UNANSWERED***: how to compute conflict sets in combinations of theories, e.g. $T = T_1 \cup T_2$*?*
- ***REMARK***: all the proposed solutions compute "***small***" but not necessarely ***minimal*** conflict sets
  - *minimal* conflict set CS = there is no $CS' \subset CS$ which is unsatisfiable in T to ensure efficiency... NP-hard for the theory of uninterpreted function symbols*!*

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Two contributions

1. Modular computation of conflict sets in combinations of theories by extending the Nelson-Oppen combination schema ***[NO79]*** via the concept of ***explanation graphs***

2. Study of the relationship between the computed “small” conflict sets and minimal ones via the concept of ***quasi-conflict set***

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Explanation graph

- Intuition: a labelled graph (V, E) where
  - V is a set of constants (occurring in the input set of literals)
  - E is an elementary equality (i.e., an equality between constants)
  - labels are explanations of elementary equalities
- An explanation EX for an elementary equality $x = y$ is such that
  - EX is satisfiable in the background theory T
  - $T \cup EX \models x = y$
- An explanation EX for an elementary equality $x = y$ is minimal iff there is no $EX' \subset EX$ which is an explanation for $x = y$

Property

An explanation EX is minimal for $x = y$ iff $EX \cup \{x \neq y\}$ is a minimal conflict set.

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An explanation graph at work: congruence closure

Init

$$\Omega; E; G \vdash$$
$$\Omega; E; \mathit{Insert}(G, z = z', \{z = t, z' = t\})$$
$$\text{if } \begin{cases} z = t, z' = t \in \Omega \\ z \neq z', (z, z') \notin CP(G) \end{cases}$$

Ins

$$\Omega; E \cup \{x = x'\}; G \vdash$$
$$\Omega; E; \mathit{Insert}(G, x = x', \{x = x'\})$$
$$\text{if } x \neq x', (x, x') \notin CP(G)$$

Skip

$$\Omega; E \cup \{x = x'\}; G \vdash$$
$$\Omega; E; G$$
$$\text{if } x = x' \text{ or } (x, x') \in CP(G)$$

An explanation graph at work: congruence closure (cont'd)

Cong $\Omega; E; G \vdash$
$\Omega; E; \mathit{Insert}(G, z = z', \{z = t, z' = t'\} \cup \bigcup_{j \in J} \{y_j = y'_j\})$

$$\text{if} \begin{cases} z = t \equiv f(y_1, \ldots, y_n), z' = t' \equiv f(y'_1, \ldots, y'_n) \in \Omega \\ z \neq z', (z, z') \notin CP(G) \\ I, J \text{ is a partition of } \{1, \ldots, n\} \text{ such that } J \neq \emptyset \text{ and} \\ (\forall i \in I : y_i = y'_i), (\forall j \in J : (y_j, y'_j) \in CP(G)) \end{cases}$$

Similar to the approach (based on proof forests) in ***[NO05]***...

Nelson-Oppen schema *[NO79]* with explanation graphs

- ***GOAL***: compute conflict sets for $T_1 \cup T_2$ when
  - the theory T_i is convex, stably-infinite, for which a satisfiability procedure is available
  - T_1 and T_2 are signature-disjoint
- ***Intuition***: the Nelson-Oppen method consists in exchanging entailed elementary equalities between the two procedures until...
- ***Key idea***: the unsatisfiability in $T_1 \cup T_2$ can be explained according to two kinds of explanations:
  1. the explanation of entailed elementary equalities
  2. the explanation of the unsatisfiability in a component theory

  To combine these explanations, the key idea is to use specialized satisfiability procedures (called ***explanation engines***) with the capability of generating explanation graphs in order to store entailed equalities

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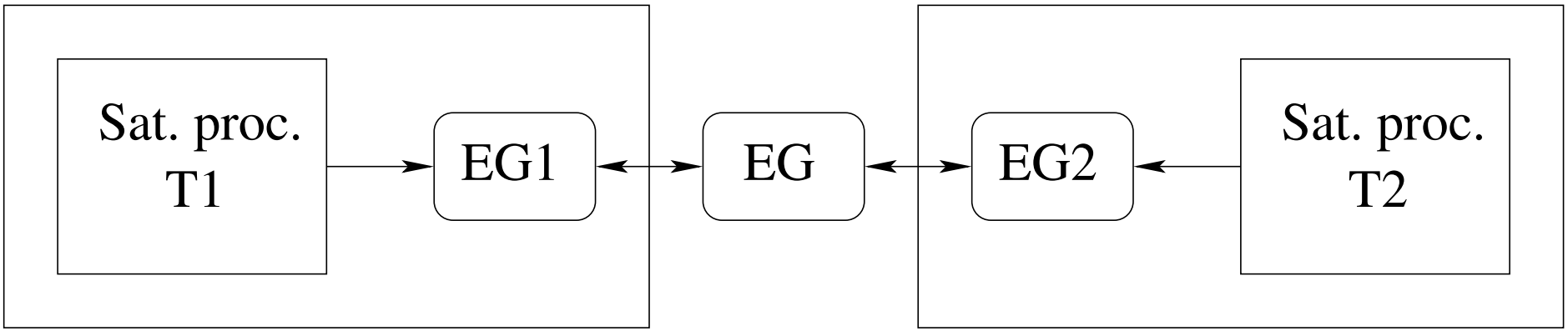
Modular combination of explanation engines: idea



Modular combination of explanation engines: formally

Explanation engine $\mu EX(\Omega, E) = (\Omega', E', G)$...

$\text{Unsat}_{=1}$

$$\Omega_1; \Delta_V; G; \Omega_2 \vdash$$
$$\mathit{false}\{(\Omega'_1, E'_1, G')\}$$
$$\text{if} \begin{cases} \mu EX_1(\Omega_1, Eq(G)) = (\Omega'_1, E'_1, G_1) \;\&\; \Omega'_1 \neq \emptyset \\ G' = \mathit{Merge}(G, G_1) \end{cases}$$

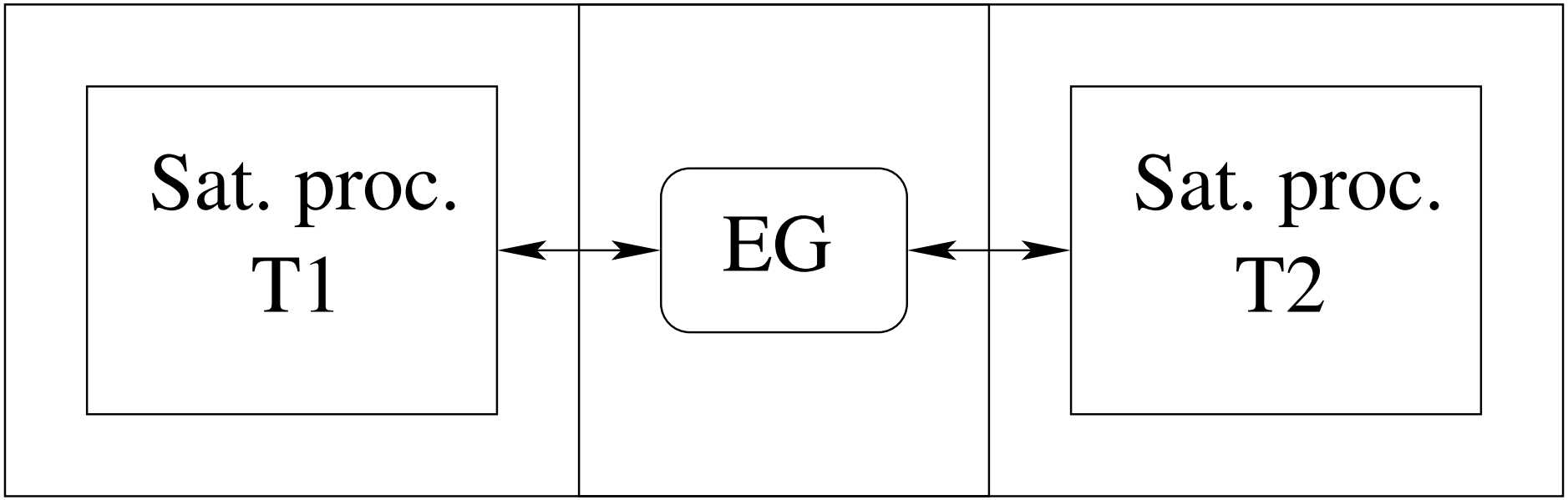
$\text{Unsat}_{\neq}$

$$\Omega_1; \Delta_V; G; \Omega_2 \vdash$$
$$\mathit{false}\{(\{x \neq y\}, \{x = y\}, G)\}$$
$$\text{if } (x, y) \in CP(G) \text{ and } x \neq y \in \Delta_V$$

Deduction_1

$$\Omega_1; \Delta_V; G; \Omega_2 \vdash$$
$$\Omega_1; \Delta_V; G'; \Omega_2$$
$$\text{if} \begin{cases} \mu EX_1(\Omega_1, Eq(G)) = (\emptyset, E_1, G_1) \\ G' = \mathit{Merge}(G, G_1) \\ G' \neq G \end{cases}$$

Modular combination of explanation engines: refinement



Quasi-conflict sets: informally

- The tuple (ψ, E, G) computed by the previous schema is such that
  - $\psi \cup E$ is unsatifiable,
  - ψ is a satisfiable subset of the input set, and
  - E is a set of (entailed) elementary equalities explained in G
- $\psi \cup E$ is not a conflict set of the input set φ since it may contain literals which are not in φ
- However, it is easy to extract a "true" conflict set from $\psi \cup E$ since E is entailed by φ and the related explanations are encoded in the associated explanation graph G
- We define the tuple (ψ, E, G) a ***quasi-conflict set*** and it is possible to define an ***ordering*** on such tuples which allows us to introduce the ***minimality of quasi-conflict sets***

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Quasi-conflict sets: basic properties

Property

If (ψ, E, G) *is a quasi-conflict set of the input set* φ *of literals, then* $\psi \cup Lit(G)$ *is a conflict set of* φ*.*

Theorem

Let (ψ, E, G) *be a quasi-conflict set of* φ *such that* $\psi \cup E$ *is a minimal conflict set. If all edges of* $G_{|E}$ *are minimally explained then* $(\psi, E, G_{|E})$ *is a minimal quasi-conflict set of* φ*.*

Discussion

- Two contributions
  1. Modular computation of conflict sets in combinations of theories by combining ***explanation engines*** (extension of the Nelson-Oppen combination schema)
  2. ***Quasi-conflict sets*** as a characterization of the minimality that can be obtained in practice when combining satisfiability procedures
- Delayed Theory Combination ***[BBC+05]*** as an alternative to avoid the computation of conflict sets in combinations of theories...

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